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THE INVERSE OF A TRIDIAGONAL MATRIX

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Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

September 1972

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THE INVERSE OF A TRIDIAGONAL MATRIX

by

Palmer R. Schlegel
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RDT&E Project No. 1T062110A027

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The closed form inverse of a fairly general tridiagonal matrix is given. The restriction is that the off-diagonal elements in the tri-diagonal band be nonzero. If the elements of the matrix are integers, where the upper off-diagonal elements are equal and the lower off-diagonal elements are equal, then an integer multiple of each element of the inverse can be generated by integer arithmetic.

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I. INTRODUCTION

The closed form inverse of a fairly general tridiagonal matrix is given. The restriction is that the off-diagonal elements in the tridiagonal band be nonzero. If the elements of the matrix are integers, where the upper off-diagonal elements are equal and the lower off-diagonal elements are equal, then an integer multiple of each element of the inverse can be generated by integer arithmetic.

In an earlier note[1*] an identity was obtained. Here we present a simpler proof of a more general relation. Multiplication of a vector by a tridiagonal matrix consists of applying a three term recurrence operation to the sequence of elements of the vector. We give the explicit expression for the inverse of the matrix in terms of two vectors, U and V, formed by recursive solution of the homogeneous relation with the zero boundary condition at the beginning and at the end respectively.

II. FORMS OF MATRIX AND ITS INVERSE

Let $B = (b_{ij})$ be a tridiagonal matrix such that $b_{i,i+1} \neq 0$ and $b_{i+1,i} \neq 0$, $i = 1,\ldots,n-1$. Define $u_0 = 0$, $u_1 = 1$ and $u_{i+1} = -b^{-1}_{i,i+1}(b_{i,i-1}u_{i-1} + b_{ii}u_i)$ for $1 \leq i \leq n-1$ and $v_{n+1} = 0$, $v_n = 1$ and $v_{j-1} = -b^{-1}_{j,j-1}(b_{jj}v_j + b_{j,j+1}v_{j+1})$ for $n+1 > j > 1$. Let

$$\lambda_j^{-1} = b_{j,j-1}v_j u_{j-1} + b_{jj}v_j u_j + b_{j,j+1}v_{j+1}u_j,$$

where

$$\begin{aligned} c_{ij} &= \lambda_j v_j u_i \ , \ i \leq j \\ &= \lambda_j v_i u_j \ , \ i > j \ , \end{aligned}$$

*Superscript numerals refer to references found on page 10.

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is the inverse of B. Conversely, if B has an inverse, it is obtained by this construction. To see this let $U^T = (u_1,\dots,u_n)$ and $V^T = (v_1,\dots,v_n)$, where u_i and v_i are defined above. Then $(BU)^T = (0,\dots,0,\xi)$ and $(BV)^T = (\eta,0,\dots,0)$, where ξ and η are some numbers. Let C_j be the jth column of B^{-1}. Since the first j-1 elements of BC_j are zero, for some r_j

$$c_{ij} = r_j u_i \ , \ i \le j.$$

Similarly, since the last n-j-1 elements of BC_j are zero, for some s_j

$$c_{ij} = s_j v_i \ , \ i \ge j.$$

Thus,

$$r_j u_j = c_{jj} = s_j v_j ,$$

or for some λ_j

$$r_j = \lambda_j v_j$$

and

$$s_j = \lambda_j u_j .$$

The jth element of BC_j is

$$\begin{aligned} 1 &= b_{j,j-1} c_{j-1,j} + b_{jj} c_{jj} + b_{j,j+1} c_{j+1,j} \\ &= b_{j,j-1} r_j u_{j-1} + b_{jj} r_j u_j + b_{j,j+1} s_j v_{j+1} \\ &= \lambda_j (b_{j,j-1} v_j u_{j-1} + b_{jj} v_j u_j + b_{j,j+1} u_j v_{j+1}), \end{aligned}$$

and this determines λ_j.

III. INTEGER ELEMENTS

Suppose the elements of B are integers. Furthermore, suppose $b_{i,i+1} = c$ and $b_{i+1,i} = d$ for $i = 1,\ldots,n-1$. Let

$$x_{i+1} = -(cdx_{i-1} + b_{ii}x_i),$$

$$y_{j-1} = -(b_{jj}y_j + cdy_{j+1})$$

and

$$w_j = cdx_{j-1}y_j + b_{jj}x_jy_j + cdx_jy_{j+1},$$

where $x_0 = 0$, $x_1 = 1$, $y_{n+1} = 0$ and $y_n = 1$. It can be shown that

$$\begin{aligned} w_jc_{ij} &= c^{j-i}x_iy_j \ , \ i \le j \\ &= d^{i-j}x_jy_i \ , \ i > j, \end{aligned}$$

that is, an integer multiple of each element of the inverse can be generated by integer arithmetic.

REFERENCES

1. P. Schlegel, "The Explicit Inverse of a Tridiagonal Matrix," Math of Comp., Vol 24, 1970, p665.